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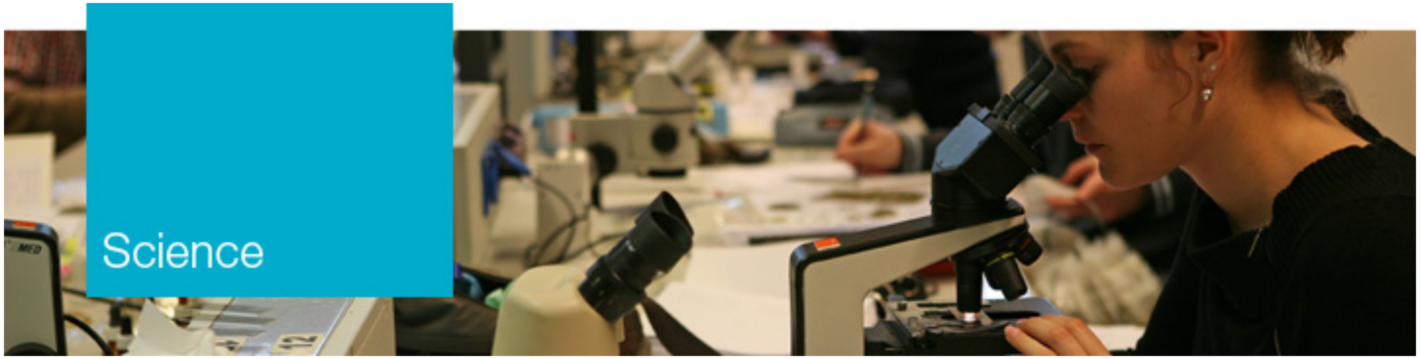

Coadjoint orbits of Poincaré group

Supervisors: N.Boulanger & A.Campoleoni

Ismael Ahlouche Lahlali

Faculty
of Science

Lie group

Definition:

A Lie group is a manifold G that has a group structure consistent with its manifold structure in the sense that group multiplication :

$$\mu : G \times G \to G, \quad (g, h) \mapsto gh$$

is a C^∞ map

Their is also the inversion map $I : G \to G; \quad g \mapsto g^{-1}$

It possed an identity element: e

Lie group can be finite or infinite dimensional

Lie group

1) SO(3)

SO(3): $\{R_{3\times 3} \mid det(R) = 1 \,\&\, R^t R = 1\}$

We can identify SO(3) as the set of rotation matrices in the 3 dimensional Euclidean space

$$R_{\mathbf{x}}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \qquad R_{\mathbf{y}}(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \qquad R_{\mathbf{z}}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2)Poincaré group:

Symmetry group of Minkowski space, describe relativistic particle, Origine of relativistic wave equations

Lie algebra

A Lie algebra is a vector space $\mathfrak{g}$ $(T_e G)$ with a binary operation $[\,.\,,.\,]: \ \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket

The Lie bracket have to satisfy the following axioms:
1)Bilinearity

$$[ax+by,z]=a[x,z]+b[y,z]$$
$$[z,ax+by]=a[z,x]+b[z,y]$$

where $x,y,z \in \mathfrak{g} \quad a,b \in \mathfrak{R}$

2)Anticommutativity
$$[x,y]=-[y,x]$$

3)Jacobi identity
$$[x,[y,z]]+[z,[x,y]]+[y[z,x]]=0$$

Lie algebra

We call generators of a Lie group, the element of the set that generate the whole Lie algebra

Exemple: SO(2)
1 parameter required
$R = 1 - i\delta\phi J$
J: generator of SO(2)

SO(3):
3 generators $\{J_x, J_y, J_z\}$

ISO(1,3):
10 generators: 4 translations, 3 boosts, 3 rotations

Important property: from the Lie algebra you can generate any FINITE element of the group by the exponential map

Representation of a group

Representation of a Lie group

A representation of a Lie group G is given by (V,ρ) where V is a vector space and $\rho : G \to GL(V)$

Casimir operator: it is an operator that commute with every elements of the representation

Exemple:

-Angular momentum in quantum mechanics

Introduction

- Symplectic geometry developed by Jean-Marie Souriau
- 1961-1962,Kirillov,Kostant find a correspondance between Orbits $\longleftrightarrow$ UIR for a given group
- Goal:Generalize the work of Kosinski

Poincare group for massive particle

- Poincaré group in d dimensions: $SO(d-1,1) \rtimes \mathbb{R}^d$
- $\dfrac{d(d+1)}{2}$ Generators P, M
- Multiplication law: $(\Lambda_1, a_1) \star (\Lambda_2, a_2) = (\Lambda_1\Lambda_2, \Lambda_1 a_2 + a_1)$
- Casimirs operators→Dimension dependent
- In D=4, $P^2 = m^2$ and $W^2 = -m^2 s(s+1)$
- Study Coadjoint orbit ⟷ Fix **every** Eigenvalue of Casimirs operator(Seen as constraints)

Unitary irreducible representation of Poincare group

Type	Orbit	Associated UIR
$p^2 = m^2$	Mass-shell	Massive
$p^2 = 0$	Light-cone	Massless
$p^2 = -m^2$	Hyperboloid	Tachyonic

For light-cone orbit 2 distincts case:

-Finite spin

~~-Infinite spin~~

Coadjoint orbits

For a given Lie group G, algebra $\mathfrak{g}$, dual algebra $\mathfrak{g}^*$

We have the pairing $\langle \eta, x \rangle \in \mathfrak{R}$ where $x \in \mathfrak{g}, \eta \in \mathfrak{g}^*$

the coadjoint action :

$\langle Ad_g^*(\eta), x \rangle = \langle \eta, g^{-1} x g \rangle$ where g $\in G$

the coadjoint orbit to a specific point η_0

$Orb(\eta_0) = \{Ad_g^*(\eta_0) \mid g \in G\}$

Coadjoint orbits

Properties of coadjoint orbits

- Coadjoint orbits of a finite Lie algebra is symplectic manifold which is even dimensional

- If G is a Lie group,$\eta \in \mathfrak{g}^*$, G_η its stability subgroup under the co-adjoint action then $Orb(\eta) = G/G_\eta$

Algorithm to find UIR

We use the following Algorithm:

- Fix a canonical point by looking at constraints

- Find a suitable parametrization of the group

- Act on this point with Ad*

- Quantize the orbit

Quantization

Step for quantization

Function mapped to linear hermitian operators

- Weyl Ordering $\longrightarrow$ $p_0 x_a \rightarrow \frac{1}{2}(\hat{p}_0\hat{x}_a + \hat{x}_a\hat{p}_0)$

- Von Newman rule:

$$\{p, x, s\} \rightarrow \{\hat{p}, \hat{x}, \hat{s}\}$$

Dual algebra of Poincaré group

$$P_\mu \rightleftharpoons \zeta_\mu$$

$$M_{\mu\nu} \rightleftharpoons \zeta_{\mu\nu}$$

$$W \rightleftharpoons \omega$$

From commutator we get Poisson brackets

$$[P_\alpha, P_\beta] = 0 \rightleftharpoons \{\zeta_\alpha, \zeta_\beta\} = 0$$

$$[M_{\mu\nu}, P_\beta] = i(\eta_{\nu\beta}P_\mu - \eta_{\mu\beta}P_\nu) \rightleftharpoons \{\zeta_{\mu\nu}, \zeta_\beta\} = \eta_{\nu\beta}\zeta_\mu - \eta_{\mu\beta}\zeta_\nu$$

$$[M_{\mu\nu}, M_{\alpha\beta}] = i(\eta_{\mu\beta}M_{\nu\alpha} + \eta_{\nu\alpha}M_{\mu\beta} - \eta_{\mu\alpha}M_{\nu\beta} - \eta_{\nu\beta}M_{\mu\alpha}) \rightleftharpoons \{\zeta_{\mu\nu}, \zeta_{\alpha\beta}\} = \eta_{\mu\beta}\zeta_{\nu\alpha} + \eta_{\nu\alpha}\zeta_{\mu\beta} - \eta_{\mu\alpha}\zeta_{\nu\beta} - \eta_{\nu\beta}\zeta_{\mu\alpha}$$

Contents

ISO(4,1)

ISO(4,1) : 15 generators

3 Casimirs:

$P^2 = m^2$, $W^{\mu\nu}W_{\mu\nu}$, $\mathbb{H} = W^{\mu\nu}M_{\mu\nu}$ Where $W^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma\delta}M_{\rho\sigma}P_{\delta}$, with $\mu \in (0,..,4)$

Such that:

$$\frac{1}{8}(W^{\mu\nu}W_{\mu\nu} + m\mathbb{H}) = m^2 s_1(s_1 + 1)$$

$$\frac{1}{8}(W^{\mu\nu\rho}W_{\mu\nu} - m\mathbb{H}) = m^2 s_2(s_2 + 1)$$

Little group contain $SO(4) = SO(3) \times SO(3)$

→Dim(O)=12

Coadjoint Orbit

The coadjoint action for Poincare group:

$$Ad_g^*(\zeta_\mu) = \Lambda_\mu^{\ \nu}\zeta_\nu$$

$$Ad_g^*(\zeta_{\mu\nu}) = \zeta_{\alpha\beta}\Lambda_\mu^{\ \alpha}\Lambda_\nu^{\ \beta} - \zeta_\beta(a_\nu\Lambda_\mu^{\ \beta} - \Lambda_\nu^{\ \beta}a_\mu)$$

where $g = (\Lambda, a) \in ISO(4,1)$

massive particle: $\zeta^\mu\zeta_\mu = m^2 \rightarrow \overline{\zeta^\mu} = (m,0,0,0,0)$

if we consider only rotation $\rightarrow \overline{\zeta_{ab}} = \bar{S}^{cd}\epsilon_{cdab}$, with (a,b) $\in (1,\ldots,4)$

looking at translation $\rightarrow Ad^*_{(1,a)}(\zeta_{0b}) = \zeta_{0b} - ma_b \rightarrow \overline{\zeta_{0b}} = 0$

Stabilizer: $SO(4) \times R$

Massive particle D=5

We can use the same parametrization as D=4 :

Lorentz elements: $\Lambda = LR$ where $R = \overline{R}B$
L:Pur boost

B $\in G_s\overline{R}$: Rotation that put s into its actual direction

$\bar{R}$ transform S

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & 0 & \sin\beta & \cos\beta \end{pmatrix}$$

Translation elements: a=z+y

$z = (z^0, \overrightarrow{0}), y = (0, \overrightarrow{y})$

$\rightarrow (\Lambda, a) = (1,y)(L,0)(\overline{R},0)(B,z)$

Massive particle D=5

Acting with Ad_g^*:

$$\zeta_{ab} = -p_a x_b + p_b x_a + J_{ab}$$

$$\zeta_{0a} = -p_0 x_a + \frac{p^b J_{ab}}{m + p^0}$$

Where: $$x_a \equiv y_a - \frac{\epsilon_{abcd} S^{bc} p^d}{(m + p^0)}, \quad J_{ab} = S^{cd} \epsilon_{abcd}$$

Poisson brackets take the form:

$$\{J^{ab}, J^{cd}\} = (\delta^{ac} J^{bd} + \delta^{bd} J^{ac} - \delta^{ad} J^{bc} - \delta^{bc} J^{ad}), \quad \{x_a, p_0\} = \frac{-p_a}{p_0}, \qquad \{p_\mu, p_\nu\} = 0$$

$$\{x_a, x_b\} = 0, \qquad \{x_a, p_b\} = \delta_{ab}, \{x_a, J^{bc}\} = 0, \qquad \{p_a, J^{bc}\} = 0$$

Representation of Poincaré group

Inner product: $(f, g) = \int \tilde{dp}\, \overline{f(p)} g(p)$

x's becomes a operator : $\hat{x}_a = +\, i(\frac{\partial}{\partial p_a} - \frac{p_a}{2|p^2|})$

Representation:

$$\hat{M}_{0a} = -\, i\hat{p}_0 \frac{\partial}{\partial p_a} + \frac{\hat{p}_d \hat{S}_{bc} \epsilon_{abcd}}{m + p_0}$$

$$\hat{M}_{ab} = \zeta_{ab} = i(\hat{p}_b \frac{\partial}{\partial p_a} - \hat{p}_a \frac{\partial}{\partial p_b}) + \hat{S}_{cd} \epsilon_{abcd}$$

Contents

Massless particle D=5

Massless particle $\rightarrow P^2 = 0 \rightarrow \zeta^\mu = (k,0,0,0,k)$

Non zero components of Pauli-Lubanski tensor:

$$\begin{aligned}
\omega^{01} &= \omega^{41} = -k\zeta_{23}, \quad \omega^{12} = k(\zeta_{30} + \zeta_{34}) \\
\omega^{02} &= \omega^{42} = -k\zeta_{31}, \quad \omega^{23} = k(\zeta_{10} + \zeta_{14}) \\
\omega^{03} &= \omega^{43} = -k\zeta_{12}, \quad \omega^{31} = k(\zeta_{20} + \zeta_{24}) \\
\omega^{04} &= 0
\end{aligned}$$

finite spin $\rightarrow \epsilon_{\mu\nu\rho\sigma\delta}\zeta^\rho\omega^{\sigma\rho} = 0$ (Helicity condition) yields to

$$\begin{aligned}
\mu = 0, \nu = 1 &\rightarrow \omega^{23} = 0 \\
\mu = 0, \nu = 2 &\rightarrow \omega^{31} = 0 \\
\mu = 0, \nu = 3 &\rightarrow \omega^{12} = 0
\end{aligned}$$

$\rightarrow \zeta_{0i} = \zeta_{i4}$

Massless particle D=5

Another parameter has to be taken in consideration!!

The spin equation (specify the UIR of the little group):

$$\mathbb{S}_\mu \equiv \epsilon_{\mu\nu\rho\sigma\delta}\zeta^{\nu\rho}\omega^{\sigma\delta} = s^2\zeta_\mu$$

in components that yield to:

$$[(\zeta_{04})^2 + (\zeta_{12})^2 + (\zeta_{31})^2 + (\zeta_{32})^2] = s^2$$

we fix our canonical point to be:

$$\overline{\zeta_\mu} = (k,0,0,0,-k)$$

$$\overline{\zeta_{ab}} = \overline{S}^{cd}\epsilon_{abcd}$$

$$\overline{\zeta_{0a}} = 0$$

Dim(O)=10

Parametrization

Lorentz elements:
(In light-cone coordinates)

$$B=\begin{pmatrix} \Lambda^+_+ & 0 & 0 & 0 & 0 \\ \Lambda^-_+ & \frac{1}{\Lambda^+_+} & \frac{\Lambda^1_+}{\Lambda^+_+} & \frac{\Lambda^2_+}{\Lambda^+_+} & \frac{\Lambda^3_+}{\Lambda^+_+} \\ \Lambda^1_+ & 0 & 1 & 0 & 0 \\ \Lambda^2_+ & 0 & 0 & 1 & 0 \\ \Lambda^3_+ & 0 & 0 & 0 & 1 \end{pmatrix} D=\begin{pmatrix} 1 & \frac{\Lambda^+_-}{\Lambda^+_+} & d_1 & d_2 & d_3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & d_1 & 1 & 0 & 0 \\ 0 & d_2 & 0 & 1 & 0 \\ 0 & d_3 & 0 & 0 & 1 \end{pmatrix} R=\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_1 & b_2 & b_3 \\ 0 & 0 & c_1 & c_2 & c_3 \\ 0 & 0 & e_1 & e_2 & e_3 \end{pmatrix}$$

Translation elements:
a=DRh+y
$D, R, h \in G_s$

Coadjoint orbit

Acting with this parametrization and performing a suitable variable change one find:

$$\zeta_{ij} = J_{ij} + p_j x_i - p_i x_j$$

$$\zeta_{4i} = p_i x_4 - p_4 x_j + \frac{p^k J_{ki}}{\sqrt{2} p^+}$$

$$\zeta_{0i} = -y_i p_0 - \frac{p^k J_{ki}}{\sqrt{2} p^+}$$

$$\zeta_\mu = p_\mu$$

The poisson takes the following form:

$$\{\zeta_\mu, \zeta_\nu\} = 0 = \{p_\mu, p_\nu\}, \quad \{\zeta_{0i}, p_0\} = -\zeta_i = -p_i, \quad \{J_{ki}, p_\mu\} = 0$$

$$\{y_i, p_0\} = \frac{p_i}{p_0}, \quad \{y_i, p_j\} = \delta_{ij}, \quad \{y_\mu, y_\nu\} = 0, \quad \{y_\mu, J_{jk}\} = 0$$

$$\{J_{kl}, J_{ij}\} = \delta_{ki} J_{lj} + \delta_{lj} J_{ki} - \delta_{kj} J_{li} - \delta_{li} J_{kj}$$

Quantization

$$\hat{M}_{04} = -ip_0 \frac{\partial}{\partial p_4}$$

$$\hat{M}_{ij} = \hat{J}_{ij} + i(p_j \frac{\partial}{\partial p_i} - p_i \frac{\partial}{\partial p_j})$$

$$\hat{M}_{0i} = -ip_0 \frac{\partial}{\partial p_i} - \frac{p^k \hat{J}_{ki}}{p_0 - p_4}$$

$$\hat{M}_{4i} = i(-p_4 \frac{\partial}{\partial p_i} + p_i \frac{\partial}{\partial p_4}) - \frac{p^k \hat{J}_{ki}}{p_0 - p_4}$$

Contents

Generalization massive particle

We can use the same parametrization and the same decomposition as D=4

Acting with this parametrization on the canonical point we get:

$$\zeta_\mu \equiv \Lambda_\mu{}^0 m = p_\mu$$

$$\zeta_{0a} = -p_0 x_a + \frac{p^e J_{ae}}{m + p^0}$$

$$\zeta_{ab} = p_b x_a - p_a x_b + J_{ab}$$

Quantization

$$\hat{M}_{0a} = +\, ip_0 \frac{\partial}{\partial p_a} + \frac{p_b \hat{J}_{ab}}{m + p_0}$$

$$\hat{M}_{ab} = i(p_b \frac{\partial}{\partial p_a} - p_a \frac{\partial}{\partial p_b}) + \hat{J}_{ab}$$

Generalization massless particle

By the same way we get:

$$\zeta_{ij} = J_{ij} + p_j x_i - p_i x_j$$

$$\zeta_{d-1i} = p_i x_{d-1} - p_{d-1} x_j + \frac{p^k J_{ki}}{\sqrt{2} p^+}$$

$$\zeta_{0i} = -y_i p_0 - \frac{p^k J_{ki}}{\sqrt{2} p^+}$$

$$\zeta_\mu = p_\mu$$

Quantization

$$\hat{M}_{0d-1} = -ip_0 \frac{\partial}{\partial p_{d-1}}$$

$$\hat{M}_{ij} = \hat{J}_{ij} + i(p_j \frac{\partial}{\partial p_i} - p_i \frac{\partial}{\partial p_j})$$

$$\hat{M}_{0i} = -ip_0 \frac{\partial}{\partial p_i} - \frac{p^k \hat{J}_{ki}}{p_0 - p_{d-1}}$$

$$\hat{M}_{d-1i} = i(-p_{d-1} \frac{\partial}{\partial p_i} + p_i \frac{\partial}{\partial p_{d-1}}) - \frac{p^k \hat{J}_{ki}}{p_0 - p_{d-1}}$$

Bibliography

[1] Krzysztof Andrzejewski, Cezary Gonera, Joanna Goner, Piotr Kosinski, and Pawel Maslanka. Spinning particles, coadjoint orbits and Hamiltonian formalism. *arXiv:2008.09478 [hep-th, physics:math-ph, physics:quant-ph]*, September 2020. arXiv: 2008.09478.

[2] Krzysztof Andrzejewski, Agnieszka Kijanka-Dec, Piotr Kosiński, and Paweł Maślanka. Chiral fermions, massless particles and Poincare covariance. *Physics Letters B*, 746:417–423, June 2015.

[3] Xavier Bekaert and Nicolas Boulanger. The unitary representations of the Poincare group in any spacetime dimension. *arXiv:hep-th/0611263*, November 2006. arXiv: hep-th/0611263.

[4] M. Chaichian, A. P. Demichev, and N. F. Nelipa. The Casimir operators of inhomogeneous groups. *Communications in Mathematical Physics*, 90(3):353–372, September 1983.

[5] Piotr Kosiński and Paweł Maślanka. Relativistic Symmetries and Hamiltonian Formalism. *Symmetry*, 12(11), 2020.

[6] Sergei M. Kuzenko and Alec E. Pindur. Massless particles in five and higher dimensions. *Physics Letters B*, 812:136020, January 2021.

[7] Jeroen Maes. An Introduction to the Orbit Method. page 103.

[8] Jerrold E Marsden. A Basic Exposition of Classical Mechanical Systems. ., page 473.

[9] Steven Weinberg. *RELATIVISTIC QUANTUM MECHANICS*, volume 1, page 49–106. Cambridge University Press, 1995.

[10] E. Wigner. On unitary representations of the inhomogeneous lorentz group. *Annals of Mathematics*, 40(1):149–204, 1939.

[11] Edward Witten. Coadjoint orbits of the Virasoro group. *Communications in Mathematical Physics*, 114(1):1–53, March 1988.