# Two-Particle Problem With Conformal And Gauge Invariance

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April 21, 2021

1. Derivation of a conformal invariant action for two massless relativistic particles
2. Study of the system using Dirac formalism
3. Breaking of the conformal invariance
4. Quantization
5. Solving the wave equation and discussion

# Single free massive particle

## Action of a free relativistic particle

$$S = -m \int d\tau \sqrt{-\dot{x}^2} \tag{1}$$

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The momenta $p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}$ are such that all velocities cannot be expressed as $\dot{x}^\mu = \dot{x}^\mu(x, p)$.

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In our case, the constraint is:

$$p^2 + m^2 \approx 0 \tag{2}$$

# Hamiltonians and action with the einbein

$S$ is invariant under reparametrization $\tau \to \tau - \varepsilon(\tau)$.
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The arbitrary function $e(\tau)$ is called einbein.
Then:

$$S_E = \int \; d\tau \; (p_\mu \dot{x}^\mu - \frac{e}{2}(p^2 + m^2)) \tag{4}$$

Using the stationary condition, we express $p_\mu$ as functions of $(x^\nu, e)$ and reaches:

## Action in the massless case

$$S = \int \; d\tau \frac{\dot{x}^2}{e} \tag{5}$$

$e$ is a variable.

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- **Poincaré transformations:** $x^\mu \longrightarrow x'^\mu = a^\mu + \Lambda^\mu{}_\nu x^\nu$
- **Dilation:** $x^\mu \longrightarrow x'^\mu = \lambda x^\mu$
- **Special conformal transformations :** $x^\mu \longrightarrow x'^\mu = \frac{x^\mu + \alpha^\mu x^2}{1 + 2\overline{\alpha}.\overline{x} + \alpha^2 x^2}$

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This is checked if the einbein transforms as:

$$e \longrightarrow \lambda^2 e \quad \text{under dilation} \tag{6}$$

$$e \longrightarrow \frac{e}{x^4} \quad \text{under inversion} \tag{7}$$

# Conformal invariant action for two particles

Goal: find action for two interacting particles.

- $x_1^\mu$: coordinates of the particle 1
- $x_2^\mu$: coordinates of the particle 2
- Potential term: function of the relative positions. We define:

$$r^\mu = x_1^\mu - x_2^\mu$$

$r^2$ transforms under inversion as:

$$r'^2 = \frac{r^2}{x_1^2 x_2^2} \tag{8}$$

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$$r^\mu = x_1^\mu - x_2^\mu$$

For two massless interacting relativistic particles, the action is:

$$S = \int d\tau \left( \frac{\dot{x_1}^2}{2e_1} + \frac{\dot{x_2}^2}{2e_2} + \frac{\alpha^2}{4} \frac{\sqrt{e_1 e_2}}{r^2} \right) \tag{9}$$

$\alpha > 0$.

# Analysis with Dirac formalism

$$\frac{\partial L}{\partial \dot{e}_i} = 0$$

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## Hamiltonians

Canonical Hamiltonian:

$$H_c = \frac{e_1 p_1^2}{2} + \frac{e_2 p_2^2}{2} - \frac{\alpha^2}{4} \frac{\sqrt{e_1 e_2}}{r^2} + \dot{e}_1 \pi_1 + \dot{e}_2 \pi_2 \qquad (10)$$

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We then find the total Hamiltonian:

$$H_T = \frac{e_1 p_1^2}{2} + \frac{e_2 p_2^2}{2} - \frac{\alpha^2}{4} \frac{\sqrt{e_1 e_2}}{r^2} + \lambda^1 \pi_1 + \lambda^2 \pi_2 \tag{11}$$

$\lambda_1$ and $\lambda_2$ are arbitrary functions.

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$\lambda_1$ and $\lambda_2$ are arbitrary functions.

Time derivative of a function:

$$\frac{d}{d\tau} f(x,p) = \{f, H_T\}$$

# Analysis with Dirac formalism

The primary constraints must be stable:

$$\{\pi_1, H_T\} = \frac{1}{2}\Big( - p_1^2 + \frac{\alpha^2}{4r^2}\sqrt{\frac{e_2}{e_1}}\Big) := \phi_1 \approx 0 \tag{14}$$

$$\{\pi_2, H_T\} = \frac{1}{2}\Big( - p_2^2 + \frac{\alpha^2}{4r^2}\sqrt{\frac{e_1}{e_2}}\Big) := \phi_2 \approx 0 \tag{15}$$

$\phi_1$ and $\phi_2$ are secondary constraints.

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$\phi_1$ and $\phi_2$ are secondary constraints. We also impose $\{\phi_i; H_T\} \approx 0$:

$$\phi_3 = \frac{\alpha^2}{16r^2}\Big( - \lambda_1\sqrt{\frac{e_2}{e_1^3}} + \lambda_2 \frac{1}{\sqrt{e_1 e_2}}\Big) \tag{16}$$

$$+ \frac{\alpha^2}{4r^4}\Big(\sqrt{e_1 e_2} p_1 . r + \sqrt{\frac{e_2^3}{e_1}} p_2 . r\Big) \tag{17}$$

# Analysis with Dirac formalism

$\phi_3$ vanishes if:

$$\lambda_1 = \lambda_2 \frac{e_1}{e_2} + \frac{4}{r^2}\left[e_1^2 p_1.r + e_1 e_2 p_2.r\right] \tag{18}$$

Thus:

## Final total Hamiltonian

$$H_T = H_c + \tilde{\lambda}(e_1\pi_1 + e_2\pi_2) + C\pi_1 \tag{19}$$

Where $C = \frac{4}{r^2} e_1 (e_1 p_1.r + e_2 p_2.r)$.

# Analysis with Dirac formalism

## Classification of the constraints

- First-class: $\phi$ is first-class if the Poisson brackets with every constraints vanishes on-shell.
- Second-class: at least one Poisson bracket with a constraint does not vanish on the constraint surface.

# Analysis with Dirac formalism

## Classification of the constraints

- First-class: $\phi$ is first-class if the Poisson brackets with every constraints vanishes on-shell.
- Second-class: at least one Poisson bracket with a constraint does not vanish on the constraint surface.

- $e_1\pi_1 + e_2\pi_2$ is a primary first-class constraint
- $H_T$ is first-class, so $\{H_T, e_1\pi_1 + e_2\pi_2\}$ too.
- Second-class constraints: $\pi_1$ and $\phi_1$

# Constraints and symmetry generator

In summary: we have two first-class (FC) constraints, a primary and a secondary, and two second-class (SC) constraints, a primary and a secondary too.

## Summary

Constraints:

$$\begin{aligned} \text{FC}: \quad & \sigma_1 = e_1\pi_1 + e_2\pi_2 \ (\text{primary}) \quad ; \quad \sigma_2 = e_1\phi_1 + e_2\phi_2 - C\pi_1 && (20) \\ \text{SC}: \quad & \pi_1 \ (\text{primary}) \quad ; \quad \phi_1 && (21) \end{aligned}$$

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Constraints:

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$$\text{SC}: \quad \pi_1 \ \text{(primary)} \quad ; \quad \phi_1 \tag{21}$$

## Symmetry generator

We can compute the symmetry generator thanks to the "chain algorithm":

$$G = \frac{d}{d\tau}(\varepsilon e_1)\pi_1 + \frac{d}{d\tau}(\varepsilon e_2)\pi_2 - (\varepsilon e_1)\phi_1 - (\varepsilon e_2)\phi_2 \tag{22}$$

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Symmetry transformations:

$$\delta e_i = \frac{d}{d\tau}(\varepsilon e_i) \qquad \delta x_i^\mu = \varepsilon \dot{x}_i^\mu \tag{23}$$

$\varepsilon(\tau)$ is a time-dependent arbitrary function.

# degrees of freedom

$$\#dof = \#\text{phase space variables} - \#\text{SC constraints} - 2.\#\text{FC constraints}$$

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## Counting of the dof

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Massless interacting case:

$$\#dof = 14$$

If we turn off the interaction:

$$\#dof = 12$$

Free case should be studied independently

# Massive case

We add a mass $m$ at the particles:

**Action in massive case**

$$S = \int d\tau \left[\frac{\dot{x}_1^2}{2e_1} + \frac{\dot{x}_2^2}{2e_2} - m^2\left(\frac{e_1 + e_2}{2}\right) + \frac{\alpha_m^2}{4}\frac{\sqrt{e_1 e_2}}{r^2}\right] \tag{24}$$

We have conformal invariance if $m = 0$.

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We have conformal invariance if $m = 0$.

Total Hamiltonian:

$$H_{Tm} = H_{T0} + \left( \frac{e_1 + e_2}{2} \right) m^2 \tag{25}$$

# Massive case

## Constraints

$$\text{FC:} \quad \sigma_{1m} \quad (primary) \quad ; \quad \sigma_{2m} \tag{26}$$
$$\text{SC:} \quad \pi_1 \quad (primary) \quad ; \quad \phi_{1m} \tag{27}$$

Such that:

$$\phi_{1m} = \phi_{1m=0} - \frac{m^2}{2} \tag{28}$$
$$\phi_{2m} = \phi_{2m=0} - \frac{m^2}{2} \tag{29}$$
$$\sigma_{1m} = e_1\pi_1 + e_2\pi_2 = \sigma_{1m=0} \tag{30}$$
$$\sigma_{2m} = e_1\phi_{1m} + e_2\phi_{2m} - C\pi_1 \tag{31}$$

# Massive case

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We count one symmetry generator:

$$G_m = \dot{\varepsilon}\sigma_{1m} + \tilde{\lambda}_2\sigma_{1m} - \sigma_{2m} \tag{34}$$

And 14 degrees of freedom in phase space:

$$\#dof = 20 - 2 - 2 \; . \; 4 = 14$$

# Killing vectors

Adding a mass at the particle breaks the conformal invariance.

## Noether charges

$$Q = \sum_i \zeta_i^\mu(x_1, x_2) p_{i\mu} \tag{35}$$

$\zeta_i^\mu(x_1, x_2)$ are the Killing vectors and $p_{i\mu} = \frac{\partial L}{\partial \dot{x}_i^\mu}$.

$\dot{Q} = 0$ imposes conditions to the Killing vectors.

# Killing vectors, massless case

## Masless case

We find that Killing vectors must obey the conformal Killing equation:

$$\partial_{i(\gamma}\zeta_{i\mu)}(x_i) = \frac{1}{4}\eta_{\gamma\mu}\partial_i^{\alpha}\zeta_{i\alpha} \tag{36}$$

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Therefore, there is conformal invariance and thus 15 independent vectors:

$$\zeta_i^\mu = a^\mu + M^{\mu\nu}x_{i\nu} + \lambda x_i^\mu + B^\nu(\eta_\nu^\mu x_i^2 - 2x_{i\nu}x_i^\mu) \tag{37}$$

## 15 conserved quantities

- Translations: $P_\mu = p_{1\mu} + p_{2\mu}$
- Rotations: $L_{[\mu\nu]} = x_{1\mu}p_{1\nu} + x_{2\mu}p_{2\nu} - x_{1\nu}p_{1\mu} - x_{2\nu}p_{2\mu}$
- Dilation: $D = x_1^\nu p_{1\nu} + x_2^\nu p_{2\nu}$
- SCT: $S_\mu = x_1^2 p_{1\mu} + x_2^2 p_{2\mu} - 2x_{1\mu}x_1^\nu p_{1\nu} - 2x_{2\mu}x_2^\nu p_{2\nu}$

# Killing vectors, massive case

In the massive case, we do not have the conformal Killing equation:

$$\partial_{i(\mu}\zeta_{i\nu)}(x_i) = 0 \tag{38}$$

The conformal symmetry is broken.

## Poincarré transformations

- Translations: $P_\mu = p_{1\mu} + p_{2\mu}$
- Rotations: $L_{[\mu\nu]} = x_{1\mu}p_{1\nu} + x_{2\mu}p_{2\nu} - x_{1\nu}p_{1\mu} - x_{2\nu}p_{2\mu}$

# Quantization, Dirac method

## Quantization of second-class constraints

- Suppose $\hat{\chi}_\alpha |\psi\rangle = 0$. Thus $[\hat{\chi}_\beta, \hat{\chi}_\alpha] |\psi\rangle = 0$.
- But the matrix $[\hat{\chi}_\beta, \hat{\chi}_\alpha]$ is invertible by definition $\implies |\psi\rangle = 0$. (We don't want this).

Thus we use the Dirac brackets in order to cancel the second-class constraints.

## Dirac brackets

$$[F; G]^* := \{F; G\} - \{F; \chi_\alpha\}(C^{-1})^{\alpha\beta}\{\chi_\beta; G\}$$

$C_{\alpha\beta} = \{\chi_\alpha, \chi_\beta\}$

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Thus we use the Dirac brackets in order to cancel the second-class constraints. These latter are mapped on a trivial operator.

## Correspondence rule

$$[\hat{A}; \hat{B}] = i\hbar \ \widehat{[A; B]}^* \tag{39}$$

$$\chi_\alpha = 0 \Rightarrow \quad \hat{\chi}_\alpha = \hat{0} \tag{40}$$

# Reduced space quantization

## Reduced space quantization

- We define gauge conditions $C_b$: after gauge fixing, any function of canonical variables can be viewed as the restriction in that gauge of a gauge invariant function.
- $det(\{C_b, FC_\alpha\}) \neq 0$.
- Quantization of FC constraints and gauge conditions $\approx$ quantization of SC constraints.

## Center of mass coordinates

$$p_+^\mu = p_1^\mu + p_2^\mu \quad (41)$$
$$p_-^\mu = p_1^\mu - p_2^\mu \quad (42)$$
$$q_+^\mu = \frac{1}{2}(x_1^\mu + x_2^\mu) \quad (43)$$
$$q_-^\mu = \frac{1}{2}(x_1^\mu - x_2^\mu) \quad (44)$$

Second-class constraints are dropped:

## FC constraints in CM coordinates with SC = 0

$$\gamma_1 = \pi_2 \approx 0$$

$$\gamma_2 = (p_+^2 + p_-^2 + 4m^2 - 2p_+.p_-)(p_+^2 + p_-^2 + 4m^2 + 2p_+.p_-) - \frac{\alpha^4}{16q_-^4} \approx 0$$

# Gauge conditions

There are two first-class constraints $\Longrightarrow$ we can fix two independent gauge conditions.

## Relevant gauge conditions

$$G_1 := p_+^\gamma p_{-\gamma} \approx 0 \qquad\qquad G_2 := e_1 e_2 - 1 \approx 0 \tag{45}$$

- $p_+ = (M, 0, 0, 0)$ and $p_- = (0, \vec{p}_-)$
- $p_1^2 = p_2^2$: equitable energy distribution
- Via EOM: $e_1 = e_2 = 1$ and $q_- = (0, \vec{q}_-)$

# Gauge conditions

In particular, the constraint $\gamma_2$ becomes:

$$(\hat{p}_+^2 + \hat{p}_-^2 + 4m^2)^2 = \frac{\alpha^4}{16\hat{q}_-^4} \tag{46}$$

Consider the operator $\hat{M}^2$:

$$\hat{M}^2 = M^2 \mathbb{1} = 4m^2 \mathbb{1} + \hat{\vec{p}}_-^{\,2} - \frac{\alpha^2}{4\hat{\vec{q}}_-^{\,2}} \tag{47}$$

A physical state is an eigenstate of $\hat{M}^2$ with the eigenvalue $\vartheta^2$:

$$\hat{M}^2 \left|\Psi\right\rangle = \vartheta^2 \left|\Psi\right\rangle \tag{48}$$

## Equation

$$\left(4m^2 - \triangle_- - \frac{\alpha^2}{4\vec{q}_-^{\,2}}\right) \Psi(q_-^\mu, q_+^\mu) = \vartheta^2 \Psi(q_-^\mu, q_+^\mu) \tag{49}$$

## Solution

$$\Psi(r,\theta,\varphi) = A\sqrt{\frac{\pi}{2r\sqrt{\vartheta^2 - 4m^2}}}\; J_\phi(\sqrt{\vartheta^2 - 4m^2}\,r) Y_{l,m_l}(\theta,\varphi) \tag{50}$$

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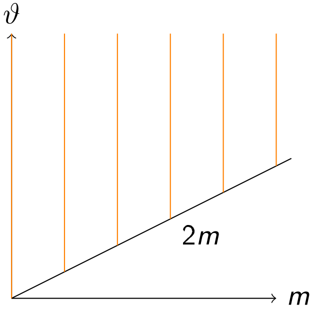

# $\alpha$ can be $r-$dependent

**Notice**

We can choose

$$\alpha(r) = \alpha_0 + \kappa f(r)$$

- No conformal invariance
- Still 2 FC constraints and 2 SC constraints. Same number of dof.
- After quantization, constraint $\gamma_2$ stays the same (except $\alpha = \alpha(r)$)!

$$\Psi_{l,m_l}(r,\theta,\varphi) = R(r)Y_{l,m_l}(\theta,\varphi) \tag{51}$$

Such that:

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + Rr^2(\vartheta^2 - 4m^2) + (\frac{\alpha^2(r)}{4} - l(l+1))R = 0 \tag{52}$$

$\rightarrow$ Discreet spectrum can be obtained.

# Summary and discussion

## Summary

- We have found a conformal invariant action for massless interacting particles.
- We can give a mass at the particles and break the conformal invariance as wished.
- We have the liberty to choose $\alpha$ in order to have a continuous or a discreet spectrum.

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## Discussion

- Interesting playground: playing with $m$ and $\alpha(r)$, we can easily go from massive confined case to conformal window with continuous spectrum.
- Unparticles: particles described with a scale invariant gauge theory with a continuous mass spectrum. $\rightarrow$ $S$ is a good candidate for unparticles.

Thank you for your attention !

# Bibliography


1 Lesson from UWaterloo (Canada): Bessel functions of the first and second kind.

2 Xavier Bekaert and Jeong-Hyuck Park. Symmetries and dynamics in constrained systems. The European Physical Journal C, 61(1):141, March 2009.

3 Roberto Casalbuoni and Joaquim Gomis. Conformal symmetry for relativistic point particles. Phys. Rev. D, 90:026001, Jul 2014.

4 Leonardo Castellani. Symmetries in constrained hamiltonian systems. Annals of Physics, 143(2):357–371, 1982.

5 Michael Eastwood. Notes on conformal differential geometry. In Proceedings of the 15th Winter School "Geometry and Physics", pages [57]–76. Circolo Matematico di Palermo, 1996.

6 Howard Georgi. Unparticle physics. Phys. Rev. Lett., 98:221601, 2007.

# Bibliography


7 J Gomis, M Henneaux, and J M Pons. Existence theorem for gauge symmetries in hamiltonian constrained systems. Classical and Quantum Gravity, 7(6):1089–1096, jun 1990.

8 Petersen Lyng Jens. Introduction to the malcedana conjecture on ads/cft. International Journal of Modern Physics A, 14(23):3597–3672, sept 1999.

9 Henneaux Marc. Quantization of gauge systems / Marc Henneaux and Claudio Teitelboim. Princeton University Press, Princeton (N.J.), C 1992.

10 H. Nikolic. Unparticle as a particle with arbitrary mass. arXiv:0801.4471 [hep-ph, physics:hep-th], July 2008. arXiv: 0801.4471 version: 3.

11 Spindel Philippe. Mécanique, volume 2, Mécanique analaytique. Contemporary Publishing International - GB Science Publichers, Paris (France), 2002.